TonyCrane 线性代数II(H)

环 ① $R\neq\emptyset$ ② $\langle R,+\rangle$ 是阿贝尔群 ③ $\langle R,\cdot\rangle$ 是半群 $(ab)c=a(bc)$

④ $a(b+c)=ab+ac$ $(b+c)a=ba+ca$

设有恒等元的环R 总能找到比它大的含恒等元的环 $\mathbb{Z}\times R$

可交换 commutative $\forall a,b\in R$ $ab=ba$. 不可交换 uncommutatile

零因子 zero-divisor $\forall b\neq 0$. 若 $a\in R$. $ab=0$ 则a为左零因子 同理右

整环 domain ① $|R|\geq 2$. $(R\neq\{0\})$ ② R没有非零的零因子.

可逆 若 $a\in R$ $\exists b\in R$ s.t. $ab=ba=1_R$ 则 a可逆.

除环 division ring ① R中任意非0元素都可逆 ② $|R|\geq 2$ $(R\neq\{0\})$

可交换除环是域

布尔环 bool ring ① $\forall a\in R$, $a^2=a$ ② R可交换

子环 substring $\emptyset\neq S\subseteq R$, $\forall a,b\in S$ ① $a-b\in S$ ② $ab\in S$

则S是R的一个子环. 记为 $S\leq R$

生成子环 $X\subseteq R$ $\langle X\rangle=\{\sum_{i_1\cdots i_s} n_{i_1}\cdots n_{i_s} x_1^{i_1}\cdots x_s^{i_s} \mid x_i\in X, i_j\in\mathbb{N}\}$

理想 ideal. $\emptyset\neq S\subseteq R$ 满足 ① $\forall a,b\in S$, $a-b\in S$.

② $\forall a\in S$. $b\in R$. $ba\in S$. 则 S为左理想 left ideal

② $\forall a\in S$ $b\in R$ $ab\in S$ 则 S为右理想 right ideal.

S既是左理想.又是右理想.则 S为理想.

生成左理想 $(a\rangle=\{ra+na \mid r\in R, n\in\mathbb{Z}\}$.

生成右理想 $\langle a)=\{ar+an \mid r\in R, n\in\mathbb{Z}\}$.

生成理想 $(a)=\{na+\sum_{i=0}^{m} r_i a s_i \mid n\in\mathbb{Z}, m\in\mathbb{N}, r_i,s_i\in R\}$

主理想 principle ideal 由一个元素生成的理想

主理想整环 PID commutative principle ideal domain

环同态 $\varphi: R_1\to R_2$ 满足 ① $\varphi(a+b)=\varphi(a)+\varphi(b)$

② $\varphi(ab)=\varphi(a)\varphi(b)$ ③若存在恒等元. 则 $\varphi(1_{R_1})=1_{R_2}$

$\forall a\neq b$, $\varphi(a)\neq\varphi(b)$ φ为单同态 monomorphism

$\forall r\in R_2$，$\exists a\in R_1$ s.t. $\varphi(r)=a$. φ为满同态、epimorphism

既单又满则φ为同构 isomorphism

同态核、$\mathrm{Ker}\,\varphi=\{a\in R_1 \mid \varphi(a)=0\}$. 是$R_1$一个理想

同态像 $\mathrm{Im}\,\varphi=\{\varphi(a)\mid a\in R_1\}$ 是R_2一个子环

商环 quotient ring R/I　$\pi: R\to R/I$　$r\mapsto r+I$

$\mathrm{Ker}\,\pi=\{a\in R \mid a+I=0+I=I\}=I$

极大理想 maximal ideal　$I\neq R$是R的理想 如果$\forall J\lhd R$

$I\subseteq J$ 有 $J=I$ 或 $J=R$. 则 I是极大理想

R是有恒等元的交换环. 则 M 是极大理想 $\Leftrightarrow$ R/M是域

素理想 prime ideal. 假设R是有恒等元的交换环. $P\neq R$是R的理想, 若 $\forall a,b\in R$, $ab\in P\Rightarrow a\in P$ 或 $b\in P$, 则P称为素理想

P是有恒等元的交换环R上素理想 $\Leftrightarrow$ R/P是整环

P是PID R的素理想. 则 $P=0$ 或 P是极大理想

素元 prime　$p\in$有恒等元交换环R. p不是零因子. 且p不可逆.

if $p\mid ab\Rightarrow p\mid a$ or $p\mid b$　$(p\mid a\Leftrightarrow a=px\Leftrightarrow (p)\supseteq(a))$

p是PID R上素元. $\Leftrightarrow$ $R_p=(p)$是极大理想

$a\in$有恒等元交换环R. 且a不是零因子也不可逆. 若 $a=bc\Rightarrow b$不可逆或c不可逆. 则称 a 不可约 (irreducible)

Thm　R是PID. $a\in R$不可约 $\Leftrightarrow$ a是素元

$f(x)\in\mathbb{Q}[x]$　$f(x)=cf_1(x)$　$c\in\mathbb{Q}$　$f_1(x)=a_nx^n+\cdots+a_1x+a_0\in\mathbb{Z}[x]$

$a_0, a_1\cdots a_n$ are coprime $\Rightarrow$ $f_1(x)$称为本原多项式 (primitive poly.)

Gauß's Lemma　$f(x)\in\mathbb{Q}[x]$ is irreducible $\Leftrightarrow$ $f_1(x)\in\mathbb{Z}[x]$ is irreducible

Eisenstein's irreducible criterion. $f(x)=a_nx^n+\cdots+a_1x+a_0\in\mathbb{Z}[x]$

$\exists p$ is prime s.t. $p\nmid a_n$　$p\mid a_i$ $0\le i\le n-1$　$p^2\nmid a_0$

then $f(x)$ is irreducible.

$$\begin{tikzcd} R_1 \arrow[rr, bend left, "\varphi"] \arrow[dr, "\pi"'] & & R_2 \\ & R_1/I \arrow[ur, "\bar{\varphi}"'] & \end{tikzcd}$$

$\varphi: R_1 \to R_2$ is a homomorphism

(1) $\mathrm{Ker}\,\varphi = \{a \in R_1 \mid \varphi(a) = 0\}$ is an ideal of R_1

(2) I 是 R_1 理想. $I \subseteq \mathrm{Ker}\,\varphi$, 则 $\exists$ 同态 $\bar{\varphi}: R_1/I \to R_2$ s.t.

$\bar{\varphi}(a+I) = \varphi(a)$ $\quad \mathrm{Ker}\,\bar{\varphi} = \{a+I \mid a \in \mathrm{Ker}\,\varphi\} = \mathrm{Ker}\,\varphi/I$

$\mathrm{Im}\,\bar{\varphi} = \mathrm{Im}\,\varphi$ $\qquad \pi(a) = a+I$.

$\bar{\varphi}$ 是单射 $\Leftrightarrow \mathrm{Ker}\,\varphi = I \Leftrightarrow \mathrm{Ker}\,\bar{\varphi} = \{0\}$ $\qquad R_1/\mathrm{Ker}\,\varphi \simeq \mathrm{Im}\,\varphi$

第一同态基本定理 $\varphi: R_1 \to R_2$ 是同态 则 $\bar{\varphi}: R_1/\mathrm{Ker}\,\varphi \to \mathrm{Im}\,\varphi$ 是同构 (iso)

第二同态基本定理 I, J 是 R 理想 $I \subseteq J$. 则

(1) $J/I = \{a+I \mid a \in J\}$ 是 R/I 理想 (2) $(R/I)/(J/I) \simeq R/J$

第三同态基本定理 S 是 R 子环. I 是 R 理想 则

(1) $S+I$ 是 R 子环 (2) I 是 $S+I$ 理想. (3) $I \cap S$ 是 S 理想.

(4) $S+I/I \simeq S/I\cap S$

$p(x)$ 不可约 $f(x) = p^n(x)g(x)$ $\quad p(x) \nmid g(x)$ 则 $\qquad$ 笛卡尔积

$F[x]/(f(x)) \simeq F[x]/(p^n(x)) \oplus F[x]/(g(x))$

模 ① M 是 Abel 群 ② R 是环 ③ $R\times M \to M$ 满足 $(r_1r_2)m = r_1(r_2m)$

$(r_1+r_2)m = r_1m + r_2m$ $\quad r(m_1+m_2) = rm_1 + rm_2$ $\Rightarrow$ M 是左 R 模 left R-module

幺模 unitial module. R 有恒等元 1_R. 且 $\forall m \in M$. $1_R \cdot m = m$

R 是除环. 则 R-module 称为 R 上向量空间 vector space

M 是 left R-module 则 M 是 right R^{op}-module

Hamilton-Caylay 定理 设 $A = (a_{ij})_{n\times n} \in M_n$ $f(\lambda) = |\lambda E - A|$ 为 A 特征多项式 则 $f(A) = 0$

子模 submodule. $\emptyset \neq N \subseteq {}_RM$ 且 $\forall x, y \in N$, $x-y \in N$, $\forall r \in R$ $x \in N$ $rx \in N$ 则 N 为 ${}_RM$ 子模. 记 $N \leq {}_RM$

直和 direct sum $\quad N_1, N_2 \leq {}_RM$. $\quad N_1 + N_2 = \{x+y \mid x \in N_1, y \in N_2\} \leq {}_RM$

若 $N_1 \cap N_2 = \{0\}$ 则 $N_1 + N_2$ 称为直和. 记作 $N_1 \oplus N_2$

商模 $N \leq M$. 定义 $M/N = \{m+N \mid m \in M\}$. 规定加法、数乘:

$(m_1+N)+(m_2+N) = (m_1+m_2)+N \quad r(m+N) = rm+N$

则 M/N 是一个左R模 称为 M 关于 N 的商模

模同态. $\varphi: {}_RM \to {}_RM'$ $\varphi(m_1+m_2) = \varphi(m_1)+\varphi(m_2)$ $\varphi(m_1m_2) = \varphi(m_1)\varphi(m_2)$

$\operatorname{Ker}\varphi = \{m \in M \mid \varphi(m) = 0\}$ 是 M 子模

$\operatorname{Im}\varphi = \{\varphi(m) \mid m \in M\}$ 是 M' 子模.

第一同态基本定理. $M/\operatorname{Ker}\varphi \simeq \operatorname{Im}\varphi$

R为域时. $\Leftrightarrow \dim M/\operatorname{Ker}\varphi = \dim \operatorname{Im}\varphi \Rightarrow \dim M = \dim\operatorname{Ker}\varphi + \dim\operatorname{Im}\varphi$

第二同态基本定理. $N \leq L \leq M$. $M/N / L/N \simeq M/L$

第三同态基本定理 $N, L \leq M$ $N+L/L \simeq N/N\cap L$

R为域时 $\Leftrightarrow \dim(N+L) - \dim L = \dim N - \dim(N\cap L)$

设M为一个左R模. $m_i \in M$. $r_i \in R$. $r_1m_1+r_2m_2+\cdots+r_nm_n$ 被称为

$\{m_1, \cdots, m_n\}$ 的一个线性组合

Thm. $X = \{m_1, \cdots, m_n\}$ $\langle X\rangle = \bigcap N$ $N \supseteq \{m_1, \cdots, m_n\}$

$\langle X\rangle = \{r_1m_1+\cdots+r_km_k \mid r_i \in R\}$ is a submodule

$\{r_1m_1+\cdots+r_km_k\}$ 也可写为 $Rm_1+Rm_2+\cdots+Rm_k$

有限生成模 finite generated module 若 $\exists\, m_1 \cdots m_k \in M$. 线性无关

$M = Rm_1+\cdots+Rm_k$ 则 M 为有限生成模

$\{m_1, \cdots m_k\}$ 称为一组基 basis. M 称为自由模 free module

$R^{(I)} = \{(a_i)_{i\in I} \mid a_i = 0 \text{ for all but finite}\}$.

$\forall i \in I$. $e_i = (a_j)_{j\in I}$ $a_j = \begin{cases} 0 & j \neq i \\ 1 & j = i \end{cases}$

自由模可以有包含不同个数元素的基 eg. $\operatorname{End}_R(R[x])$ $id_{R[x]}$ 和 $f_1(x^{2n}) = x^n$ $f_1(x^{2n+1}) = 0$ $f_2(x^{2n}) = 0$ $f_2(x^{2n+1}) = x^n$

$R^m \underset{\varphi^{-1}}{\overset{\varphi}{\rightleftarrows}} R^n$ $\{\alpha_1, \cdots \alpha_m\}\{\beta_1, \cdots, \beta_n\}$ $(\varphi(\alpha_1) \cdots, \varphi(\alpha_m)) = (\beta_1 \cdots \beta_n)A$ 同理 B

则 $AB = E_n$ $BA = E_m$ 若 R 为交换环. 则有 $m = n$ (rank)

Prop. M是有限生成R模. 则 $\exists$ epic $\varphi: R^n \to M$ s.t. $M \simeq R^n/\operatorname{Ker}\varphi$

Zorn's Lemma ① Ω is a nonempty partial order set

② $\forall a_1 \le a_2 \le \cdots \le a_n \le \cdots$ is a well-order subset (良序子集)

$\exists a \in \Omega$, s.t. $a_i \le a$ ($\forall i \in N^+$) Then there is an element $b \in \Omega$

satisfying $\forall a \in \Omega$ if $b \le a \Rightarrow b = a$ (b is maximal)

单模 simple module. ① $M \ne 0$ ② ${}_RN \le {}_RM \Rightarrow N=0$ or $N=M$

半单模 semisimple module ${}_RM = \sum_{i\in I} \oplus T_i$ T_i is simple.

Lemma 2.2.1 N是半单模 $M=\sum_{i\in I} S_i$ 的子模 (S_i 为单模) 则∃I的子集J

满足 $M = N \oplus (\sum_{i\in J} \oplus S_i)$

Thm. D是除环. 对于D上模 D^m 与 D^n. $D^m \simeq D^n \Leftrightarrow m=n$

Lemma. $S_1 \cdots S_m$ $T_1 \cdots T_n$ 为R上单模. $S_1 \oplus \cdots \oplus S_m \simeq T_1 \oplus \cdots \oplus T_n$ 则有 $m=n$

且 $S_i \simeq T_j$ (up to order)

$A = (a_{ij})_{n\times n}$ $F[x] \times F^n \to F^n$ $(f(x), \alpha) \mapsto f(A)\cdot\alpha$ F_A^n 是F[x]上模

且 F_A^n 有限生成,由 $\{e_1, \cdots, e_n\}$ (不是基)

$\varphi: F[x]^n \to F_A^n$ $(f_1(x), \cdots, f_n(x))^T \mapsto f_1(A)e_1 + \cdots + f_n(A)e_n$

$\mathrm{Im}\varphi \supseteq \{\varphi\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \mid a_i \in F\} = F^n$ epic $\mathrm{Ker}\varphi \supseteq F[x]\alpha_1 + \cdots + F[x]\alpha_n$

$(\alpha_1 \cdots \alpha_n)$ is a basis or $\ker\varphi$.

$\alpha_i = (-a_{1i}, \cdots, -a_{i-1,i}, x - a_{ii}, -a_{i+1,i}, \cdots, -a_{ni})^T$

Lemma. R is a PID. $\Omega = \{Ra \mid a \in B\} \ne \emptyset$ then there is a maximal element in Ω. (Zorn引理)

Lemma. $M = Rx_1 + \cdots + Rx_n$ $\{x_1, \cdots, x_n\}$ is a basis R is a PID.

$\forall a = r_1x_1 + \cdots + r_nx_n \ne 0$ then M has a basis $\{z_1, \cdots, z_n\}$

s.t. $a = dz$ where $d \in R$ (归纳)

自由模的子模还自由

Thm. $0 \ne N \le M = Rx_1 + \cdots + Rx_n$. $\{x_1, \cdots, x_n\}$ is a basis. R is a PID.

There is a basis $\{y_1, \cdots, y_n\}$ $r_i \in R$ $r_i \mid r_{i+1}$ $(i = 1, 2, \cdots, m-1)$

$\Leftrightarrow Rr_{i+1} \le Rr_i$ s.t. $\{r_1y_1, \cdots, r_my_m\}$ is a basis of N $(m \le n)$

Thm. $A=(a_{ij})_{m\times n}$ $a_{ij}\in R$ R is a PID $\exists$ invertible matrix U, V

s.t. $UAV=\begin{pmatrix} d_1 & & & & \\ & \ddots & & & \\ & & d_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0\end{pmatrix}$ $d_i \mid d_{i+1}$ $1\le i\le r-1$

找这样的U.V只需初等行、列变换. 且在Z上不能用某列来除 ($E_{i(c)}$)

Prop. d is a GCD $\Leftrightarrow$ $Ra_1+\cdots+Ra_m=Rd$

Cor. $0\ne(a_1,\cdots,a_n)\in\mathbb{Z}^n$ and $GCD(a_1,\cdots,a_n)=1$ then $\exists\{y_1,\cdots,y_n\}$ basis

s.t. $y_1=(a_1,\cdots,a_n)$

Cor. $N\le R^n$ $R^n/N\simeq R/(a_1)\oplus\cdots\oplus R/(a_m)\oplus R^{n-m}$ $a_i \mid a_{i+1}$

Thm. R is PID. $\forall\, 0\ne a\in R$. $Ra\subsetneq R$ then ① $\exists\, p_1\cdots p_r\in R$ is prime,

s.t. $a=p_1p_2\cdots p_r$ ② if $a=p_1\cdots p_r=q_1\cdots q_s$ $p_1\cdots p_r\, q_1\cdots q_s$ is prime

then $r=s$ $\forall p_i\ \exists q_j$ s.t. $p_i=uq_j$ (u is unit)

中国剩余定理. I. $A_1\cdots A_r$ are ideals of commutative ring R with 1_R

and $I=A_1\cap A_2\cap\cdots\cap A_r$ $A_i+A_j=R\ (i\ne j)$

then $R/I\simeq R/A_1\oplus R/A_2\oplus\cdots\oplus R/A_r$ and $I=A_1\cap\cdots\cap A_r=A_1A_2\cdots A_r$

$(A_1A_2\cdots A_r=\{\sum_{i=1}^{r} r_1^i\cdots r_s^i \mid r_j^i\in A_{i_j}\})$

Prop. p, q are primes in a PID R and $Rp\ne Rq \Leftrightarrow \nexists\, u$ s.t. $p=uq$

finite generated ${}_RM$. R is a PID then $M\simeq R/Rr_1\oplus\cdots\oplus R/Rr_s\oplus R^t$

$=R/Rp_1^{n_1}\oplus\cdots\oplus R/Rp_k^{n_k}\oplus R^t$

$\forall\, I$ is ideal of R. $\sqrt{I}=\{a\in R \mid a^n\in R\}$

Lemma. R is a PID. $p\in R$ is prime $\forall n\ge1$ $R/Rp^n=R/(p^n)\ne A\oplus B$

where $A\ne0$ $B\ne0$

Krull-Schmidt Thm. $M=R/(p_1^{r_1})\oplus\cdots\oplus R/(p_k^{r_k})=R/(q_1^{s_1})\oplus\cdots\oplus R/(q_l^{s_l})$

$p_i\ q_j$ are primes. then $k=l$ $Rp_i^{r_i}=Rq_i^{s_i}$ up to order.

$\Leftrightarrow r_i=s_i$ $p_i=u_iq_i$ u_i is unit.

称 $p_1^{r_1}\cdots p_k^{r_k}$ 为 $tor(M)$ 的初等因子 (elementary factors)

$\mathrm{tor}(M)=\{m\in M \mid \exists\, a\neq 0,\ a\cdot m=0\}$

$A=(a_{ij})_{m\times n}$ $\quad F[x]\times F_A^n\to F_A^n$ $\quad F_A^n=F[x]/(d_1(x))\oplus\cdots\oplus F[x]/(d_r(x))$

$d_i(x)\mid d_{i+1}(x)$ $\quad d_i(x)=p_1^{e_{i1}}\cdots p_r^{e_{ir}}$ $\quad F[x]/(d_i(x))\simeq\sum_{j=1}^{r}\oplus F[x]/(p_j(x)^{e_{ij}})$

$d_1(x),\dots d_r(x)$ 为不变因子 (invariant factors)

不变→初等：中国剩余定理. 初等→不变：先取最高次，再次高 …

e.g. $M=\mathbb{Z}/(4)\oplus\mathbb{Z}/(3)\oplus\mathbb{Z}/(8)\oplus\mathbb{Z}/(25)\oplus\mathbb{Z}/(125)=\mathbb{Z}_{4\times 8\times 125}\oplus\mathbb{Z}_{3\times 25}$

$A=(a_{ij})_{m\times n}$ $\quad a_{ij}\in F$ $\quad F^n$ is a $F[x]$-module

$\varphi: F[x]^n\to F^n$ epic $\quad \varphi\begin{pmatrix}f_1(x)\\ \vdots\\ f_n(x)\end{pmatrix}=\sum_{i=1}^n f_i(A)e_i$

$\psi: F[x]^n\to F[x]^n$ $\quad \begin{pmatrix}f_1\\ \vdots\\ f_n\end{pmatrix}\mapsto (xE-A)\begin{pmatrix}f_1\\ \vdots\\ f_n\end{pmatrix}$

$F^n\simeq F[x]^n/\mathrm{Ker}\,\varphi=F[x]^n/\mathrm{Im}\,\psi$ $\quad xE-A=P(x)\,\mathrm{diag}(1,\cdots,1,d_1(x),\dots,d_s(x))\,Q(x)$

$F^n\simeq F[x]/(d_1(x))\oplus\cdots\oplus F[x]/(d_s(x))$

Cor. $d_s(A)=0$ ($\Rightarrow$ Hamilton-Caylay Thm.)

$F=\mathbb{C}$ (algebraically closed field) $p_i(x)$不可约 $\Leftrightarrow p_i(x)=x-a_i$

$F^n\simeq F[x]/((x-a_1)^{n_1})\oplus\cdots\oplus F[x]/((x-a_r)^{n_r})$

令 $e_{i1}=(\underbrace{0,\cdots,0}_{i-1},1+((x-a_i)^{n_i}),0\cdots 0)$

$e_{i2}=(0,\cdots,0,(x-a_i)+((x-a_i)^{n_i}),0\cdots 0)$

$e_{ij}=(0,\cdots,0,(x-a_i)^j+((x-a_i)^{n_i}),0,\cdots 0)$ $\quad 0\le j\le n_i-1$

$e_{11}\cdots e_{1n_1},e_{21}\cdots e_{2n_2},\cdots e_{r1}\cdots e_{rn_r}$ linearly indep.

$n_1+n_2+\cdots+n_r=n$ $\quad\hookrightarrow$ basis

$x(e_{11}\cdots e_{1n_1}\cdots e_{r1}\cdots e_{rn_r})$

$xe_{i1}=x+(x-a_i)^{n_i}F[x]=a_i+(x-a_i)+(x-a_i)^{n_i}F[x]=a_ie_{i1}+e_{i2}$

$xe_{il}=a_i(x-a_i)^{l-1}+(x-a_i)^l+(x-a_i)^{n_i}F[x]=a_ie_{i(l-1)}+e_{il}$

$$x(e_{i1}\cdots e_{in_i})=(e_{i1}\cdots e_{in_i})\begin{pmatrix}a_i&&&&\\1&a_i&&&\\&1&\ddots&&\\&&\ddots&a_i&\\&&&1&a_i\end{pmatrix}$$ ← Jordan square

$$x(e_{11}\cdots e_{1n_1}\cdots e_{r1}\cdots e_{rn_r}) = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_r \end{pmatrix} \quad J_i = \begin{pmatrix} a_i & & & \\ 1 & a_i & & \\ & 1 & \ddots & \\ & & \ddots\ 1 & a_i \end{pmatrix}$$

in $\mathbb{R}$ 不可约: ① $x-a$ ② x^2+ax+b, $a^2-4b<0$

仍为 Jordan 矩阵

$\mathbb{R}[x]/(x^2+ax+b)\mathbb{R}[x]$ 二维线性空间 $\{\bar{1}, \bar{x}\}$ 为基

$$x(\bar{1}, \bar{x}) = (\bar{x}, \overline{x^2}) = (\bar{x}, \overline{-(ax+b)}) = (\bar{x}, -a\bar{x}-\bar{b}) = (\bar{1}, \bar{x})\begin{pmatrix} 0 & -b \\ 1 & -a \end{pmatrix}$$

in $\mathbb{Q}$ 任意次均可能不可约

$\mathbb{Q}[x]/(x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n)\mathbb{Q}[x]$ 基 $\{\bar{1}, \bar{x}, \cdots, \overline{x^{n-1}}\}$

$$x(\bar{1}, \bar{x}, \cdots, \overline{x^{n-1}}) = (\bar{x}, \overline{x^2}, \cdots, \overline{x^n}) = (\bar{x}, \cdots, \overline{x^{n-1}}, \overline{-(a_1x^{n-1}+\cdots+a_0)})$$

$$= (\bar{x}, \cdots, \overline{x^{n-1}}, -\Sigma a_i \bar{x_i})$$

$$= (\bar{1}, \bar{x}, \cdots, \overline{x^{n-1}})\begin{pmatrix} 0 & & & & -a_n \\ 1 & \ddots & & & \vdots \\ & \ddots & \ddots & & \vdots \\ & & & 0 & -a_2 \\ & & & 1 & -a_1 \end{pmatrix}$$

← rational similar canonical form 有理相似标准形

选基. $F[x]/(x^n+a_1x^{n-1}+\cdots+a_n)^m F[x]$

$$e_{ij} = x^i(x^n+a_1x^{n-1}+\cdots+a_n)^j + (x^n+a_1x^{n-1}+\cdots+a_n)^m F[x]$$

$$0\le i\le n-1 \qquad 0\le j\le m-1$$

e.g. $A=\begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$ $\quad xE-A \to \begin{pmatrix} 1 & & \\ & 1 & \\ & & (x-1)(x^2+1) \end{pmatrix}$

in $\mathbb{R}/\mathbb{Q}$. $F_A^3 \simeq F[x]/(x-1) \oplus F[x]/(x^2+1)$ $\quad x-1 \to (1) \qquad x^2+1 \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

then rational canonical form of A is $\begin{pmatrix} 1 & & \\ & 0 & -1 \\ & 1 & 0 \end{pmatrix}$

in $\mathbb{C}$ $\quad F_A^3 \simeq F[x]/(x-1) \oplus F[x]/(x-i) \oplus F[x]/(x+i)$

then Jordan canonical form of A is $\begin{pmatrix} 1 & & \\ & i & \\ & & -i \end{pmatrix}$